NOV. 5TH

Displacement Current

$$V_{emf}^{tr} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{s}$$

$$= \oint_c \vec{E}\cdot d\vec{l} \leftarrow \text{time dependence now.}$$

NOTA BENE

N.B. In electrostatics: $\oint_c \vec{E}\cdot d\vec{l} = 0$

Applying Stoke's Theorem

$\vec{\nabla}\times\vec{E} = 0$ in e/statics.

Now, for our case, $\oint \vec{E}\cdot d\vec{l} \neq 0 = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{s}$ (1)

$\vec{\nabla}\times\vec{E} \neq 0$

V_{emf} is <u>not</u> localized for the time-dependent case.

Using Stokes' Theorem

$$\oint \vec{E}\cdot d\vec{l} = \int_s (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{s}$$

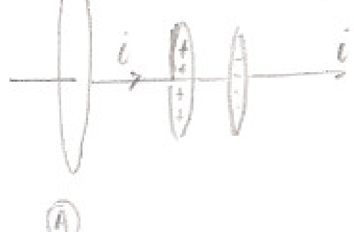
$$\Rightarrow 0 = \int_s \left(\vec{\nabla}\times\vec{E} - \frac{\partial\vec{B}}{\partial t}\right) d\vec{s}$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = \frac{\partial\vec{B}}{\partial t}$$

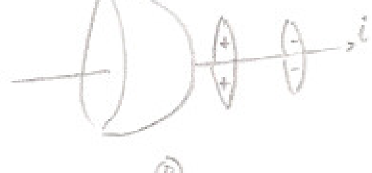
$\frac{\partial\vec{B}}{\partial t} = 0$ for static case.

The time-dependent $\vec{B}$-field induces an $\vec{E}$-field.
If a time dependent $\vec{B}$ can induce $\vec{E}$, can a time dependent $\vec{E}$ induce $\vec{B}$? $\Rightarrow$ Yes.

$\vec{H}\cdot d\vec{l} = I_{encl.}$



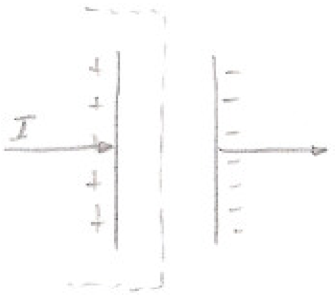
(A)



(B)

In case B) the surface encloses no current, but encloses electric flux. This flux will change with time and this results in $\vec{B}$.

Idealisations:



- no fringing
- $\vec{E}$ is constant between plates.

$$|\vec{E}| = \frac{Q(t)}{\varepsilon_0 A} \iff \frac{\sigma A}{\varepsilon A}$$

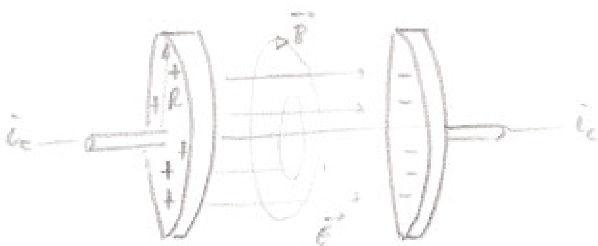
$$\Phi_E = \oint \vec{E}\cdot d\vec{s} = |\vec{E}|\cdot A = \frac{Q}{\varepsilon}$$

$$\frac{d}{dt}\Phi_E = \frac{1}{\varepsilon}\underbrace{\frac{dQ}{dt}}_{\text{current}} = \frac{I_d}{\varepsilon} \leftarrow \text{displacement current.}$$

↳ 0, if $\frac{d\Phi_E}{dt} = 0$.

We must modify Ampere's law to account for the displacement current.

$$\Rightarrow \boxed{\int \vec{H}\cdot d\vec{l} = \underbrace{I_c}_{\text{cond-current}} + \underbrace{\varepsilon \frac{d\Phi_E}{dt}}_{\text{disp-current}}}$$



$$\int (\vec{\nabla}\times\vec{H})\cdot d\vec{s} = \int \vec{J}_c\cdot d\vec{s} + \frac{d}{dt}\int \vec{D}_d\cdot d\vec{s}$$

$$\Phi_E = \int \vec{E}\cdot d\vec{s}$$

$$\varepsilon\Phi = \int \varepsilon\vec{E}\cdot d\vec{s} = \int \vec{D}\cdot d\vec{s}$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_c + \frac{\partial}{\partial t}\vec{D}$$

$\vec{D} = \varepsilon\vec{E}$ in vacuum, $\vec{D} = \varepsilon_0\vec{E}$

$\vec{J} = \sigma\vec{E}$ in a conductor.

$\varepsilon_0 \approx 10^{-12}$, $\sigma = 10^{7}$ (Copper)

For most conductors, $I_d \ll I_c$; opposite for insulators.

$\vec{\nabla} \cdot \vec{D} = \rho_s$ $\qquad\qquad$ $\vec{\nabla} \times \vec{E} = \dfrac{-\partial \vec{B}}{\partial t}$

$\vec{\nabla} \cdot \vec{B} = 0$ $\qquad\qquad$ $\vec{\nabla} \times \vec{H} = \vec{J_c} + \dfrac{\partial \vec{D_d}}{\partial t}$

Applications : waves.

Wave: disturbance in space and time, that carries energy.

Special case of 1-d waves, the mathematical description of the wave is any function of the form $\phi^- = f(x-ut)$ or $\phi^+ = g(x+ut)$, u: some constant.

ϕ^- describes a wave moving to the right (if $u>0$)
Pick some value of $t = t_0$.

$f(x_0 - ut_0)$ is max. $\qquad\qquad$ $f(x_1 - ut_1)$ is max, $\quad t_1 = t_0 + ut$

$$f(x_0 - ut_0) = f(x_1 - ut_0 - u\Delta t)$$

$$\Leftrightarrow \quad x_0 = x_1 - u\Delta t$$
$$x_1 = x_0 + u\Delta t$$

ϕ^+ describes a wave moving to the left ($u>0$)

Specialised case: free propagation in a lossless medium, f, g must satisfy

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} = 0$$

$$\frac{\partial^2 g}{\partial x^2} - \frac{1}{u^2}\frac{\partial^2 g}{\partial t^2} = 0$$

$\rightarrow$ For this specialised case, $f(x-ut) = A\cos(x-ut) + B\sin(x-ut)$

$$g(x+ut) = C\cos(x+ut) + D\sin(x+ut)$$

u: velocity of wave (phase velocity)

We will write $f(x-ut) = A\cos(\underbrace{\beta(x-ut)}_{\text{unitless}})$, $\beta \Rightarrow$ units of $\frac{1}{m}$ (m^{-1})

$$\sim A\cos(\beta x - \beta u t)$$

$\beta = \frac{2\pi}{\lambda}$, λ is the wavelength

$\beta u = \omega$, ω is the angular frequency. [Hz]

Electromagnetic waves in vacuum.

$\rho_s = 0$ ← no sources of charge
$J_c = 0$ ← no cond. current.

$\vec{D} = \varepsilon_0 \vec{E}$, $\vec{H} = \frac{\vec{B}}{\mu_0}$

"curl of $\vec{B}$"

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = \frac{\partial}{\partial t}(\mu_0 \vec{\nabla} \times \vec{H}) = \mu_0 \frac{\partial}{\partial t} \underbrace{\vec{\nabla} \times \vec{H}}_{\frac{\partial}{\partial t}\vec{D}}$$

$$\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = \mu_0 \frac{\partial^2}{\partial t^2}\vec{D} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E}$$

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{E})}_{\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{0}) - \nabla^2 \vec{E}} = -\mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E}$$

$(\vec{\nabla} \cdot \vec{D}) = \rho_s = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = 0$

$$-\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \Leftrightarrow \underbrace{\nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0}_{\text{wave equation.}}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{1}{c^2} = \mu_0 \varepsilon_0$$

In a similar way:

$$\boxed{\nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B}}$$

For the $\hat{x}$ component.

$$\frac{\partial^2}{\partial x^2} E_x + \frac{\partial^2}{\partial y^2} E_x + \frac{\partial^2}{\partial z^2} E_x - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E_x = 0$$